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Phase Diagram and Critical Properties of the Asymmetric Mattis Model

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We extend the well-known Mattis model to the case of asymmetric bond distributions. Although the partition function is identical with that of the pure ferromagnetic Ising model (FIM) when the external field is absent, the response to the external field is nontrivial even at zero field. There are some exact relations between the present model and the FIM in the correlation functions, from which the phase diagram and critical exponents can be determined. Multicritical behavior and some other interesting phenomena typical for a random system are demonstrated by this model.

KEY WORDS: Mattis model; random system; spins glass; $\pm J$ model; correlation function; critical exponent; universality.

1. INTRODUCTION

It has been a fascinating subject in physics for a long time to study the effects of randomness on cooperative systems. One of the most interesting problems is the spin-glass problem.[1-3] Some recent theoretical developments in this field originate from investigations of the $\pm J$ Ising model in finite dimensions,[4-12] which is regarded as the typical model of spin glasses. Although an important problem, the existence of a spin-glass phase at finite temperatures, has been confirmed positively in three dimensions and negatively in two dimensions for symmetric bond distributions in the $\pm J$ model,[6-9] there are some unclarified problems in the asymmetric case,[5,10-12] such as the universality and the weak universality of ferromagnetic and/or spin-glass critical exponents, the shape of the ferromagnetic phase boundary at low temperatures, the existence and the

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nature of the random antiphase state or the mixed phase, and so on. Although there are many investigations in this field, reliable results are scarce, since theoretical treatments of this model are complicated due to the randomness of the bond configuration as well as the frustration.

It is useful for understanding the thermodynamic and critical behaviors in random systems and for checking the validity of methods to analyze these systems if a solvable random model is introduced. Mattis proposed a simple solvable model with random exchange interactions,[13] called the Mattis model, which has a symmetric bond distribution containing no frustrated plaquettes. As an analog for the asymmetric $\pm J$ model, we extend the (symmetric) Mattis model to the case of asymmetric bond distributions, which we call the asymmetric Mattis model (AMM). Although the Mattis model has been understood to be insufficient for a model of real spin glasses because its dynamical behavior is too simple due to the lack of frustration, there is some nontrivial behavior in the response to an external field even at zero field. Moreover, in the AMM, one can demonstrate exactly multicritical behavior and some interesting phenomena expected in such random systems.

In the next section, we define the AMM. In Section 3 analytic properties for the free energy and the correlation functions are derived. In Section 4 we derive the phase diagram and critical exponents using the results in Section 3. Concluding remarks are given in the last section.

2. DEFINITION OF THE ASYMMETRIC MATTIS MODEL

The Hamiltonian we consider is

$$\mathcal{H}\{S\}\{J\} = -\sum_{\langle ij \rangle} J_{ij} S_i S_j \qquad (S_i = \pm 1) \tag{2.1}$$

where $\langle ij \rangle$ runs over the nearest-neighboring pairs on a d-dimensional hypercubic lattice with N sites. Each bond J_{ij} takes a value $+J$ or $-J$ randomly. The thermal average of a physical quantity at inverse temperature β for one particular bond configuration $\{J\}$ is given by

$$\langle \cdots \rangle_{\{J\},\beta} = \frac{\sum_{\{S\}} \cdots \exp(-\beta \mathcal{H}\{S\}\{J\})}{\sum_{\{S\}} \exp(-\beta \mathcal{H}\{S\}\{J\})} \tag{2.2}$$

where

$$\sum_{\{S\}} \equiv \sum_{S_1 = \pm 1} \sum_{S_2 = \pm 1} \cdots \sum_{S_N = \pm 1}$$

For the Mattis model,[13] the bond J_{ij} is determined by

$$J_{ij} = J\tau_i\tau_j \tag{2.3}$$

as there are no frustrated plaquettes in the lattice. The dimensionless random variable τ_i associated with the ith site takes a value $+1$ or -1. In the symmetric case, in which the concentration of ferromagnetic bonds p is $1/2$, each variable τ_i is selected independently. Thus, the bond average is expressed as

$$[\cdots] = \frac{1}{2^N} \sum_{\{\tau\}} \cdots \tag{2.4}$$

where

$$\sum_{\{\tau\}} \equiv \sum_{\tau_1 = \pm 1} \sum_{\tau_2 = \pm 1} \cdots \sum_{\tau_N = \pm 1}$$

In the present paper, the average in terms of the spin variables $\{S\}$ is denoted by the brackets $\langle\cdots\rangle$, and that of $\{\tau\}$ (or $\{J\}$) by $[\cdots]$.

Let us define the concentration of ferromagnetic bonds for a bond configuration $\{J\}$,

$$c\{J\} \equiv \frac{2}{zN} \sum_{\langle ij \rangle} \frac{J_{ij} + J}{2J} \tag{2.5}$$

where z is the coordination number, which is equal to $2d$ for a d-dimensional hypercubic lattice. Note that the bond average (2.4) is defined similar to a canonical ensemble in the sense that $c\{J\}$ amounts to $1/2$ as an average. Thus, the bond average (2.4) is denoted by $[\cdots]_{1/2}^{(\mathrm{M,c})}$; the first letter of the superscript, M, comes from the name of the model, and the second letter, c, comes from the "canonical" way. In each bond configuration, $c\{J\}$ is not strictly $1/2$ and has a fluctuation of order $1/\sqrt{N}$ when the system size N is finite. If one defines the bond average in a microcanonical way as

$$[\cdots]_p^{(\mathrm{M,m})} = \frac{\sum_{\{\tau\}} \cdots \delta_{c\{J\}, p}}{\sum_{\{\tau\}} \delta_{c\{J\}, p}} \tag{2.6}$$

where $\delta_{a,b}$ is the Kronecker delta, then $c\{J\}$ is strictly equal to p in each bond configuration. Due to the usual arguments for the "equivalence of ensembles[14,15]", it is reasonable to expect that the two different ways of bond averaging, $[\cdots]_{1/2}^{(\mathrm{M,c})}$ in Eq. (2.4) and $[\cdots]_p^{(\mathrm{M,m})}$ in Eq. (2.6) with $p = 1/2$, become identical in the thermodynamic limit $N \to \infty$. Therefore, we

regard $[\cdots]_p^{(\mathrm{M,m})}$ as the bond average of the AMM hereafter. We remark that not all values of p between 0 and 1 can be realized if N is finite; the possible values are ordered discretely in intervals of order $(zN)^{-1}$, whose lengths vanish when $N \to \infty$.

The extension of the symmetric Mattis model to the asymmetric case is complicated, since it is not unique. There are several ways different from Eq. (2.6). One may ask why we choose $[\cdots]_p^{(\mathrm{M,m})}$. To answer this question, we consider the asymmetric $\pm J$ model. The expressions for the Hamiltonian and the thermal average are identical with the Mattis model, Eqs. (2.1)–(2.2). For the symmetric case, the configuration average of bond randomness is given by

$$[\cdots]_{1/2}^{(\pm J,\mathrm{c})} = \frac{1}{2^{zN/2}} \sum_{\{J\}} \cdots \tag{2.7}$$

where

$$\sum_{\{J\}} \equiv \cdots \sum_{J_{ij} = \pm J} \cdots$$

It is straight forward to extend Eq. (2.7) to the asymmetric case. Since each bond J_{ij} is an independent random variable, one can define the distribution function of J_{ij} as

$$P_J(J_{ij}; p) = p\delta(J_{ij} - J) + (1 - p)\,\delta(J_{ij} + J) \tag{2.8}$$

in which the symmetric case is recovered when $p = 1/2$. Using Eq. (2.8), one may rewrite the bond average (2.7) by

$$[\cdots]_p^{(\pm J,\mathrm{c})} = \int \cdots \rho_p^{(\pm J)}\{J\} \prod_{\langle ij \rangle} dJ_{ij} \tag{2.9}$$

with

$$\rho_p^{(\pm J)}\{J\} = \prod_{\langle ij \rangle} P_J(J_{ij}; p) \tag{2.10}$$

Thus, setting $p \neq 1/2$, one obtains the bond average of the asymmetric $\pm J$ model

$$[\cdots]_p^{(\pm J,\mathrm{c})} = \sum_{\{J\}} p^{zNc\{J\}/2}(1 - p)^{zN(1 - c\{J\})/2} \cdots \tag{2.11}$$

Similarly to the case of the AMM, the bond average in a microcanonical way can be defined as

$$[\cdots]_p^{(\pm J,\mathrm{m})} = \frac{\sum_{\{J\}} \cdots \delta_{c\{J\},p}}{\sum_{\{J\}} \delta_{c\{J\},p}} \tag{2.12}$$

It is plausible that the configuration average $[\cdots]_p^{(\pm J,\mathrm{c})}$ in Eq. (2.9) and $[\cdots]_p^{(\pm J,\mathrm{m})}$ in Eq. (2.12) become identical in the thermodynamic limit.

It is easy to see that the expression for the bond average in the microcanonical way for the AMM, Eq. (2.6), is a natural extension of that for the asymmetric $\pm J$ model, Eq. (2.12). The only difference between them is the variable taken in the summation.

It is possible to define the bond average of the symmetric Mattis model instead of Eq. (2.4) using the distribution function of the local variable such as Eqs. (2.8)–(2.10),

$$P_\tau(\tau_i; 1/2) = \tfrac{1}{2}\delta(\tau_i - 1) + \tfrac{1}{2}\delta(\tau_i + 1) \tag{2.13}$$

With the distribution function of a bond configuration $\{\tau\}$,

$$\rho_{1/2}^{(\mathrm{M})}\{\tau\} = \prod_i P_\tau(\tau_i; 1/2) \tag{2.14}$$

the "canonical" bond average is expressed as

$$[\cdots]_{1/2}^{(\mathrm{M,c})} = \int \cdots \rho_{1/2}^{(\mathrm{M})}\{\tau\} \prod_i d\tau_i \tag{2.15}$$

For the asymmetric case, using the distribution function

$$P_\tau(\tau_i; p') = p'\delta(\tau_i - 1) + (1 - p')\,\delta(\tau_i + 1) \tag{2.16}$$

instead of Eq. (2.13), one can find that p' does not equal the concentration of ferromagnetic bonds p, but is related to it by $p = p'^2 + (1 - p')^2$. Then, Eq. (2.15) with Eqs. (2.14) and (2.16) is another possible version of the AMM instead of Eq. (2.6). Nevertheless, we use Eq. (2.6) as the distribution of the AMM, since the effects of randomness in Eq. (2.16) are so simple that there are no phase transitions for intermediate concentrations of p', whereas some nontrivial behavior exists as a function of p in Eq. (2.6), as will be shown later.

The essential difference between them is related to the so-called (spin)-reversal symmetry; the distribution function (2.6) gives the same probability if one transforms τ_i to $-\tau_i$ for all i, whereas such symmetry is broken in Eq. (2.16) except for $p' = 1/2$. Thus, in the case of Eq. (2.16), the correlation $[\tau_0\tau_r](= [\tau_0][\tau_r] = (2p' - 1)^2)$ is always strictly positive even if the distance r goes to infinity. Note that, in general, the ground state of Eq. (2.1) with Eq. (2.3) for a fixed $\{\tau\}$ has two degenerate spin configurations, which are identical with $\{\tau\}$ and its reversed state. Therefore, the above argument means that the ground state is always ferromagnetic for Eq. (2.16).

3. ANALYTIC PROPERTIES OF THE ASYMMETRIC MATTIS MODEL

For simplicity, the magnitude of the exchange interaction J appearing in Eqs. (2.3), (2.8), and (2.12) is set to unity hereafter. Using the function

$$e\{\tau\} \equiv -\frac{2}{zN} \sum_{\langle ij \rangle} \tau_i \tau_j \tag{3.1}$$

one may rewrite the bond average (2.6) by

$$[\cdots]_p^{(\mathrm{M,m})} = \frac{\sum_{\{\tau\}} \cdots \delta_{e\{\tau\},1-2p}}{\sum_{\{\tau\}} \delta_{e\{\tau\},1-2p}} \tag{3.2}$$

The function $e\{\tau\}$ can be regarded as the per-bond energy of the pure ferromagnetic Ising model (FIM),

$$\mathscr{H}_{\mathrm{F}}\{\tau\} = -\sum_{\langle ij \rangle} \tau_i \tau_j \qquad (\tau_i = \pm 1) \tag{3.3}$$

if $p \geqslant 1/2$, and that of the pure antiferromagnetic Ising model,

$$\mathscr{H}_{\mathrm{AF}}\{\tau\} = \sum_{\langle ij \rangle} \tau_i \tau_j \qquad (\tau_i = \pm 1) \tag{3.4}$$

if $p \leqslant 1/2$. Therefore, $[\cdots]_p^{(\mathrm{M,m})}$ in Eq. (2.6) or (3.2) describes the "microcanonical" average with the Hamiltonian $\mathscr{H}_{\mathrm{F}}\{\tau\}$ if $p \geqslant 1/2$ or $\mathscr{H}_{\mathrm{AF}}\{\tau\}$ if $p \leqslant 1/2$. Since the thermodynamic behaviors are almost identical if the concentration p is changed to $1-p$, we treat only the region $p \geqslant 1/2$ hereafter. We believe that if N is large enough, the microcanonical average (3.2) for the FIM is equivalent to the canonical average[14,15] given by

$$[\cdots]_{\beta_p}^{(\mathrm{F,c})} \equiv \frac{\sum_{\{\tau\}} \cdots \exp(-\beta_p \mathscr{H}_{\mathrm{F}}\{\tau\})}{\sum_{\{\tau\}} \exp(-\beta_p \mathscr{H}_{\mathrm{F}}\{\tau\})} \tag{3.5}$$

where the effective inverse temperature β_p is determined by

$$e_{\mathrm{F}}(\beta_p) \equiv \lim_{N \to \infty} \frac{2}{zN} [\mathscr{H}_{\mathrm{F}}\{\tau\}]_{\beta_p}^{(\mathrm{F,c})} = 1 - 2p \tag{3.6}$$

Thus, we regard the average $[\cdots]_{\beta_p}^{(\mathrm{F,c})}$ as a "canonical" bond average for the AMM, denoted by $[\cdots]_p^{(\mathrm{M,c})}$. Clearly, $[\cdots]_p^{(\mathrm{M,c})}$ with $p = 1/2$ ($\beta_p = 0$) is equivalent to $[\cdots]_{1/2}^{(\mathrm{M,c})}$ in Eq. (2.4).

The partition function of the Hamiltonian (2.1) is given by

$$Z\{J\} \equiv \sum_{\{S\}} \exp(-\beta \mathscr{H}\{S\}\{J\}) \tag{3.7}$$

It follows that $Z\{J\}$ is invariant under gauge transformations, that is, if the spin variables $\{S\}$ are transformed by any set $\{\sigma\}$ via

$$S_i \to S_i \sigma_i \tag{3.8}$$

where σ_i takes a value $+1$ or -1. Setting $\{\sigma\}$ to $\{\tau\}$ in Eq. (3.8) and performing the transformation to Eq. (3.7), one obtains

$$Z\{J\} = \sum_{\{S\}} \exp\left(\beta \sum_{\langle ij \rangle} S_i S_j\right) \equiv Z_{\mathrm{F}} \tag{3.9}$$

where Z_{F} is the partition function of the FIM. The averaged free energy is identical with that of the FIM irrespective of the concentration p,

$$f(\beta, p) \equiv -\beta [\log Z\{J\}]_p^{(\mathrm{M,m})} = f_{\mathrm{F}}(\beta) \tag{3.10}$$

Therefore, the thermodynamic properties, the behaviors of the energy, entropy, and specific heat, the critical temperature, and so on are equivalent to those of the FIM.

Note that the above statement breaks down if an external field is applied. It is expected that the response to the external field or the thermodynamic properties in the external field are different and complicated. To analyze these properties, we consider two kinds of correlation functions

$$g(r; \beta, p) \equiv [\langle S_0 S_r \rangle_{\{J\},\beta}]_p^{(\mathrm{M,m})} \tag{3.11}$$

and

$$\tilde{g}(r; \beta, p) \equiv [\{\langle S_0 S_r \rangle_{\{J\},\beta}\}^2]_p^{(\mathrm{M,m})} \tag{3.12}$$

The behavior of the correlation function $g(r; \beta, p)$, called the ferromagnetic correlation function, indicates the phase transition between ferromagnetic and nonferromagnetic phases. The behavior of $\tilde{g}(r; \beta, p)$, called the spin-glass correlation function, indicates the phase transition between paramagnetic and ordered phases, including ferromagnetic, antiferromagnetic, and spin-glass phases. By performing the transformation (3.8) with $\{\sigma\}$ set to $\{\tau\}$, we have

$$\begin{aligned} \langle S_0 S_r \rangle_{\{J\},\beta} &\equiv \sum_{\{S\}} S_0 S_r \exp\left(\beta \sum_{\langle ij \rangle} S_i S_j \tau_i \tau_j\right) \Big/ Z\{J\} \\ &= \tau_0 \tau_r g_{\mathrm{F}}(r; \beta) \end{aligned} \tag{3.13}$$

where $g_F(r;\beta)$ is the correlation function of the FIM, defined by

$$g_F(r;\beta) \equiv \langle S_0 S_r \rangle_{F,\beta} \tag{3.14}$$

The average $\langle \cdots \rangle_{F,\beta}$ denotes the thermal average with the FIM Hamiltonian $\mathscr{H}_F\{S\}$, which is equivalent to $[\cdots]_\beta^{(F,c)}$ with the variables $\{\tau\}$ replaced by $\{S\}$. Substituting Eq. (3.13) into (3.12), we obtain

$$\tilde{g}(r;\beta,p) = \{g_F(r;\beta)\}^2 \tag{3.15}$$

Since the "microcanonical" bond average $[\cdots]_p^{(M,m)}$ is identical to $[\cdots]_p^{(M,c)}$ or $[\cdots]_{\beta_p}^{(F,c)}$ in the thermodynamic limit, we have

$$\begin{aligned} g(r;\beta,p) &= g_F(r;\beta)[\tau_0\tau_r]_p^{(M,m)} \\ &= g_F(r;\beta)\, g_F(r;\beta_p) \end{aligned} \tag{3.16}$$

using Eqs. (3.11) and (3.13). Consequently, the thermodynamic behavior of the AMM can be explained by the FIM through the free energy $f_F(\beta)$ and the correlation function $g_F(r;\beta)$.

Note that it is possible to generalize the above procedure to any quantity such as $A\{S\}\{\tau\}$. The averaged quantity is expressed as

$$[\langle A\{S\}\{\tau\}\rangle_{\{J\},\beta}]_p^{(M,m)} = [\langle A\{S\tau\}\{\tau\}\rangle_{F,\beta}]_{\beta_p}^{(F,c)} \tag{3.17}$$

where $\{S\tau\}$ is a set with element $S_i\tau_i$ on the ith site.

4. PHASE DIAGRAM AND CRITICAL PROPERTIES

Let us define two kinds of order parameter,

$$\{m(\beta,p)\}^2 \equiv \lim_{r\to\infty} g(r;\beta,p) \tag{4.1}$$

and

$$\{q(\beta,p)\}^2 \equiv \lim_{r\to\infty} \tilde{g}(r;\beta,p) \tag{4.2}$$

In general, the squared magnetization $\{m(\beta,p)\}^2$ has a nonzero value in the ferromagnetic (FM) phase, and so does the spin-glass order parameter $\{q(\beta,p)\}^2$ in any ordered phase, such as the FM, antiferromagnetic, or spin-glass phase.[11] Thus, both order parameters have nonzero values in the FM phase and vanish in the paramagnetic (PM) phase. In the present model, there exists an extra phase called the Mattis spin-glass (MSG) phase, in which $\{m(\beta,p)\}^2 = 0$ and $\{q(\beta,p)\}^2 > 0$. When $d \geqslant 2$, there is a

finite critical temperature T_c in the FIM, and the following asymptotic forms are expected for the correlation function:

$$g_{\mathrm{F}}(r;\beta)\sim\begin{cases}\dfrac{\exp\{-r/\xi_{\mathrm{F}}(\beta)\}}{r^{d-2+\eta_{\mathrm{F}}}} & (T>T_c)\\[2ex] \dfrac{1}{r^{d-2+\eta_{\mathrm{F}}}} & (T=T_c)\\[2ex] \{m_{\mathrm{F}}(\beta)\}^2+\dfrac{\exp\{-r/\xi_{\mathrm{F}}(\beta)\}}{r^{d-2+\eta_{\mathrm{F}}}} & (T<T_c)\end{cases} \tag{4.3}$$

The function $\xi_{\mathrm{F}}(\beta)$ is the correlation length, and η_{F} denotes the critical exponent for the FIM (similar notations, α_{F}, $\beta_{\mathrm{F}}^{(\mathrm{e})}$, γ_{F}, and ν_{F}, are also used for other exponents). Using Eqs. (3.15), (3.16), and (4.3), we have

$$\{m(\beta,p)\}^2=\{m_{\mathrm{F}}(\beta)\}^2\{m_{\mathrm{F}}(\beta_p)\}^2 \tag{4.4}$$

and

$$\{q(\beta,p)\}^2=\{m_{\mathrm{F}}(\beta)\}^4 \tag{4.5}$$

The resulting phase diagram for $d\geqslant 2$ is shown in Fig. 1 in the p–T plane. The critical concentration p_c is determined by the condition

$$\beta_{p_c}=\beta_c \tag{4.6}$$

where $\beta_c\equiv 1/k_{\mathrm{B}}T_c$, or, equivalently,

$$e_{\mathrm{F}}(\beta_c)=1-2p_c \tag{4.7}$$

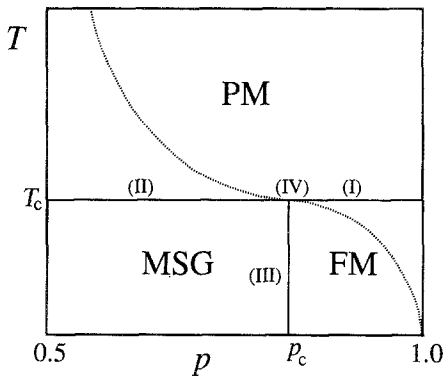

Fig. 1. The phase diagram of the AMM in the p–T plane for $d\geqslant 2$. There are three kinds of phases, the paramagnetic (PM), the ferromagnetic (FM), and the Mattis spin-glass phases. Four kinds of critical regimes are indicated (I–IV). The dotted line indicates Eq. (4.8). In the case $d=2$, one may regard the scale of this diagram as $T_c=2.269...$ and $p_c=0.853....$

In Fig. 1, the dotted line indicates the relation

$$e_{\mathrm{F}}(\beta) = 1 - 2p \tag{4.8}$$

which intersects the multicritical point (T_c, p_c), similar to the Nishimori line for the asymmetric $\pm J$ model,

$$\exp(-2\beta) = \frac{1-p}{p} \tag{4.9}$$

in which J is set to unity. Note that the average per-bond energy, $1-2p$, on the Nishimori line is identical to Eq. (4.8).

We analyze the critical behavior by using Eqs. (3.10), (3.15), and (3.16). Four kinds of critical regime are indicated in the phase diagram of Fig. 1:

(I) $T \sim T_c$ and $p > p_c$
(II) $T \sim T_c$ and $p < p_c$
(III) $T < T_c$ and $p \sim p_c$
(IV) $T \sim T_c$ and $p \sim p_c$ (the multicritical point)

From Eq. (3.10), the specific heat diverges at $\beta = \beta_c$ with the exponent $\alpha = \alpha_{\mathrm{F}}$. Let us denote by $\beta^{(\mathrm{e})}$, γ, ν, and η the exponents associated with the ferromagnetic correlation function $g(r; \beta, p)$ (the ferromagnetic critical exponents), and by $\tilde{\beta}^{(\mathrm{e})}$, $\tilde{\gamma}$, $\tilde{\nu}$, and $\tilde{\eta}$ those associated with the spin-glass correlation function $\tilde{g}(r; \beta, p)$ (the spin-glass critical exponents). We define these exponents using the scaling variable $|\beta - \beta_c|$ in the critical regimes I, II, and IV, and $|\beta_p - \beta_c|$ in III and IV; the exponents at the multicritical point IV do not change if one chooses $|\beta - \beta_c|$ or $|\beta_p - \beta_c|$ as the scaling variable. The exponents η and $\tilde{\eta}$ are defined just at the critical point itself and independent of the scaling variable, whereas the other exponents depend on it. It should be remarked that the exponents, except for η and $\tilde{\eta}$, on III and IV change if the scaling variable $|p - p_c|$ is used instead of $|\beta_p - \beta_c|$. Noting that the derivative of the energy in terms of the temperature is equal to the specific heat, one has

$$|p - p_c| \sim \left| \frac{\partial e_{\mathrm{F}}(\beta_p)}{\partial \beta_p} \right| |\beta_p - \beta_c| \sim |\beta_p - \beta_c|^{1-\alpha_{\mathrm{F}}} \tag{4.10}$$

Thus the exponent, for example, ν, changes to $\nu/(1-\alpha_{\mathrm{F}})$ if one uses $|p - p_c|$ as the scaling variable. To see the effect of the thermal and the disorder fluctuations on the phase transitions symmetrically, we redraw the phase diagram in the β_p–β plane (see Fig. 2).

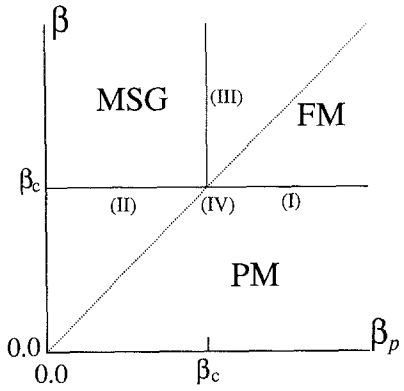

Fig. 2. The symmetric phase diagram of the AMM in the β_p–β plane.

The exponents $\beta^{(e)}$ and $\tilde{\beta}^{(e)}$ are determined from Eqs. (4.4) and (4.5) and listed in Table I. If we define the correlation lengths associated with the correlation functions $g(r; \beta, p)$ and $\tilde{g}(r; \beta, p)$ as

$$\xi(\beta, p) \equiv \lim_{r \to \infty} \left| \frac{\partial}{\partial r} \ln g(r; \beta, p) \right|^{-1} \tag{4.11}$$

and

$$\tilde{\xi}(\beta, p) \equiv \lim_{r \to \infty} \left| \frac{\partial}{\partial r} \ln \tilde{g}(r; \beta, p) \right|^{-1} \tag{4.12}$$

Table I. Critical Exponents of the AMM for Critical Regions I–IV[a]

	I	II	III	IV
α	α_F	α_F	—	α_F
$\tilde{\beta}^{(e)}$	$2\beta_F^{(e)}$	$2\beta_F^{(e)}$	—	$2\beta_F^{(e)}$
$\tilde{\nu}$	ν_F	ν_F	—	ν_F
$\tilde{\eta}$	$d-2+2\eta_F$	$d-2+2\eta_F$	—	$d-2+2\eta_F$
$\tilde{\gamma}$	$2\gamma_F - d\nu_F$	$2\gamma_F - d\nu_F$	—	$2\gamma_F - d\nu_F$
$\beta^{(e)}$	$\beta_F^{(e)}$	—	$\beta_F^{(e)}$	$2\beta_F^{(e)}$
ν	ν_F	—	ν_F	ν_F
η	η_F	—	η_F	$d-2+2\eta_F$
γ	γ_F	—	γ_F	$2\gamma_F - d\nu_F$

[a] The exponent α is determined from the exact relation (3.10). The exponents $\beta^{(e)}\nu$, η, $\tilde{\beta}^{(e)}$, $\tilde{\nu}$, and $\tilde{\eta}$ are determined from the exact relations (3.15) and (3.16). The exponents γ and $\tilde{\gamma}$ are obtained by the scaling relations. The dashes indicate that corresponding quantities are analytic and do not show singularities in these regimes.

the exponents ν and $\tilde{\nu}$ are determined as in Table I using Eqs. (3.15) and (3.16). From Eqs. (3.15) and (4.3), we derive the critical behavior of the spin-glass correlation function on the line $T = T_c$,

$$\tilde{g}(r; \beta, p) \sim 1/r^{2(d-2+\eta_F)} \tag{4.13}$$

Using Eq. (3.16) with Eq. (4.3), we obtain the critical behavior of the ferromagnetic correlation function,

$$g(r; \beta, p) \sim \{m_F(\beta_p)\}^2/r^{d-2+\eta_F} \tag{4.14}$$

for $T = T_c$ and $p > p_c$ (on the line I),

$$g(r; \beta, p) \sim \{m_F(\beta)\}^2/r^{d-2+\eta_F} \tag{4.15}$$

for $T < T_c$ and $p = p_c$ (on the line III), and

$$g(r; \beta, p) \sim 1/r^{2(d-2+\eta_F)} \tag{4.16}$$

for $T = T_c$ and $p = p_c$ (at the point IV). The resulting critical exponents η and $\tilde{\eta}$ are listed in Table I. Assuming the scaling relations

$$2 - \eta_F = \gamma_F/\nu_F \tag{4.17a}$$

$$2 - \eta = \gamma/\nu \tag{4.17b}$$

$$2 - \tilde{\eta} = \tilde{\gamma}/\tilde{\nu} \tag{4.17c}$$

one can determine the exponents γ and $\tilde{\gamma}$ as listed in Table I.

The above results are consistent with the scaling relations

$$\alpha_F + 2\beta_F^{(e)} + \gamma_F = 2 \tag{4.18a}$$

$$\alpha + 2\beta^{(e)} + \gamma = 2 \tag{4.18b}$$

$$\tilde{\alpha} + 2\tilde{\beta}^{(e)} + \tilde{\gamma} = 2 \tag{4.18c}$$

and the hyperscaling relations

$$2 - \alpha_F = d\nu_F \tag{4.19a}$$

$$2 - \alpha = d\nu \tag{4.19b}$$

$$2 - \tilde{\alpha} = d\tilde{\nu} \tag{4.19c}$$

except for those on the line III. The reason for this inconsistency on III is left unsolved. As for the ferromagnetic critical exponents, the universality (constancy of $\beta^{(e)}$, γ, and ν) and the weak universality (constancy of $\beta^{(e)}/\nu$,

γ/ν, and η) hold along the boundary of the FM phase, I and III, except for the multicritical point. The universality and weak universality for the spin-glass critical exponents hold along the boundary of the PM phases, I and II, including the multicritical point.

It is instructive to demonstrate the above results for the case $d=2$, in which there exists an exactly solved free energy and some highly reliable critical exponents in the FIM. The energy of the FIM is given as

$$e_F(\beta) = -\coth 2\beta \times \left\{ 1 + \frac{2}{\pi} (2 \tanh^2 2\beta - 1) \times \int_0^{\pi/2} \frac{d\omega}{\{1 - (2 \sinh 2\beta / \cosh^2 2\beta)^2 \sin^2 \omega\}^{1/2}} \right\} \tag{4.20}$$

Using Eqs. (4.7) and (4.20), we evaluate the critical concentration, $p_c = 0.853...$. The phase diagram in the p–T plane is identical to Fig. 1. Noting that $\alpha_F = 0$, $\beta_F^{(e)} = 1/8$, $\gamma_F = 7/4$, $\nu_F = 1$, and $\eta_F = 1/4$, we construct the list of exponents in Table II.

In one dimension, the phase transition occurs at zero temperature in the FIM. The tangent line of Eq. (4.8) does not become parallel to the p axis in finite temperatures. It is noted that the asymmetric Mattis model is identical to the $\pm J$ model in one dimension. Since the per-bond energy in the FIM is $-\tanh \beta$, the line (4.8) is equivalent to the Nishimori line (4.9). The correlation function can be calculated as

$$[\tau_0 \tau_r]_p^{(M,m)} = (2p-1)^r = \tanh^r \beta_p \tag{4.21}$$

Table II. The critical Exponents in Two Dimensions

	I	II	III	IV
α	0	0	—	0
$\beta^{(e)}$	1/4	1/4	—	1/4
$\tilde{\nu}$	1	1	—	1
$\tilde{\eta}$	1/2	1/2	—	1/2
$\tilde{\gamma}$	3/2	3/2	—	3/2
$\beta^{(e)}$	1/8	—	1/8	1/4
ν	1	—	1	1
η	1/4	—	1/4	1/2
γ	7/4	—	7/4	3/2

5. CONCLUDING REMARKS

We defined the asymmetric Mattis model by extending the symmetric one proposed by Mattis. Although the asymmetrization is not unique, the present model is a natural extension of the asymmetric $\pm J$ model. In this model, there are some nontrivial behaviors in the response to the external field even at zero field. The phase diagram and the critical exponents in d dimensions are determined in Figs. 1 and 2 and Table I. These results may be useful references to the critical phenomena in random systems such as the $\pm J$ model. The present results are derived for hypercubic lattices; however, it is easy to apply them to the AMM on other lattices. As is apparent from the phase diagram Figs. 1 and 2 and Table I, the role of the thermal fluctuation in the critical behavior is not equivalent to that of the disorder. The properties of universal critical exponents in the present model are similar to those in the $\pm J$ model.[10,11] Moreover, this model is one of the simplest references for checking a method one needs to apply to more compiicated random systems.

Using the transformation (3.8), it is found that the symmetric Mattis model with uniform fields $-h \sum_i S_i$ is equivalent to the random-field Ising model with independent binary distributions[16–18]

$$\mathscr{H} = -J \sum_{\langle ij \rangle} S_i S_j - h \sum_i \tau_i S_i \tag{5.1}$$

where the variables $\{\tau\}$ for the random fields obey the distribution (2.15) with Eqs. (2.13) and (2.14). In the case of the AMM with uniform fields, another kind of random field Ising model results, which has the same Hamiltonian as Eq. (5.1) with the distribution (2.6) or (3.5). In this model, it is clear from Eq. (3.5) that the average of each local field $[h\tau_i]_p^{(\mathrm{M,m})}$ vanishes. The distribution is not independent in the sense that

$$[(h\tau_i)(h\tau_j)]_p^{(\mathrm{M,m})} \neq [h\tau_i]_p^{(\mathrm{M,m})} [h\tau_j]_p^{(\mathrm{M,m})} \quad (=0) \tag{5.2}$$

for any i and j. It has been confirmed that the lower critical dimension of the random field Ising model is two.[16–19] Therefore, for the AMM with uniform fields, some differences in the thermodynamic properties are expected between $d=2$ and $d \geqslant 3$.

When $p > 1/2$, in about half of the disorder configurations the local fields are predominantly positive, and in the other half predominantly negative. These are the configurations for the two possible p' values [see Eq. (2.16)] corresponding to one p-value. If one uses Eq. (2.16) for the distribution of τ_i, the model is equivalent to a nonzero average random-field Ising model with independent distributions ($[h\tau_i] \neq 0$ and $[(h\tau_i)(h\tau_j)] = [h\tau_i][h\tau_j]$).

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